Chain-Necklace Diagrams with the Sum of Two Consecutive Numbers a Perfect Cube

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Abstract – **In this paper, we have used Rathore's method for arranging numbers 1-*n* such that the sum of two consecutive numbers is a perfect cube. The full chain is obtained for *n* = 305 and full necklace is obtained for *n* = 473. Larger the value of *n*, it takes more effort and time to find out chain-necklace diagram.**

Keywords: Magic circle, Chain, Necklace, Trio

I. INTRODUCTION

AS far as we know, there is no published paper on the topic arranging the numbers on a circle such that the sum of two neighbouring numbers is a perfect square or cube. For the case of square, a circle of 32 is widely circulated [1]. There are chats on a group of mathematicians and scattered results are available. Some results are available in these chats; however, no proof is openly available. It is mentioned that they have developed the method using the graph theory and/or computer programs; but not available in the open literature [1, 2].

We use the terms: *Chain*: a string of numbers satisfying certain criterion. *Full chain*: When the chain contains all the numbers. *Necklace:* When the two ends of the chain are joined. *Full necklace*: When the necklace has all the numbers.

Case A: For the sum as perfect square
1. Number 15 is the minimal for a full chain.
2. Number 32 is the minimal for a full necklace.
3. Numbers 32 and 33 have the unique full necklaces.
4. Numbers above 33 have multiple full necklaces.
5. The number of full necklaces increases rapidly as the number increases.

Case B: For the sum as a perfect cube
1. Number 305 is minimal for a full chain.
2. Number 388 claimed to be minimal; but it is not true.
3. Number 473 is minimal for full necklace.
4. The number of full necklaces increases rapidly as the number increases.

Case A has been dealt in detail in [3]. In this paper, Rathore's trio-based method [3] is extended to case B. It is simple and systematic. Unlike graph theory, it can be used even by a school child. Using the method, we prove all the above properties for case B.

A *trio* (T) is a set of 3 numbers (a,b,c) such that $a+b$ and $b+c$ satisfy some given criterion and b has no other choice except a and c. For example, in case A (B) criterion is perfect square (cube).

Some of the properties which are useful are summarized for easy implementation of the method.

1) If a T (a,b,c) is such that $a + c$ is also a perfect square, then (a,c,b) is also a T. It means b and c can be interchanged.
2) If two strings (Ss) are $\{a,b,c,d, \ldots x,y,z\}$ and $\{c,d,\ldots,x,y,z\}$, then latter can be ignored.
3) Merging property: Two Ss $\{a,b,\ldots,p,q,v\}$ and $\{p,q,v,..,w,x,y,z\}$ can be merged in to a single S $\{a,b,\ldots,p,q,v,..w,x,y,z\}$.
4) Mixing property: If two Ss are $\{a,b,c,d,\ldots,h\}$ and $\{i,j,\ldots w,x,y,z\}$ are such that $h + i$ is a perfect cube, they can be replaced by a single S $\{a,b,c,d,\ldots,h,i,,j,\ldots.,w,x,y,z\}$.

Properties 2 and 3 will be helpful in eliminating the common numbers present between the two Ss. Property 4 will be useful in interconnecting the Ss.

II. THE METHOD

The Rathore's trio-based method [1] is extended to find the chains and necklaces when the sum of two consecutive numbers is a perfect cube. The method consists of the following steps.

1) To prepare a table for given n for the choices of various natural numbers.
2) To prepare a list of all the Ts from Table 1.
3) To eliminate the common numbers from various Ts using the properties 2 and 3. This will result in several Ss.
4) To list the left-out numbers (LONs) out of Ts.
5) To insert the LONs around the two ends of the strings such that the criterion laid down should not be violated.
6) To interconnect the various Ss, such that the criterion laid down should not be violated, using property 4.

To demonstrate the working of the method, we take specific values of n = 296, 300, 305, 386 and 473.

Table 1-- Choices FOR NUMBERS 1-296

1-7,26,63,124,215,	26-1,38,99,190	51-13,74,165,292	76-49,140,267	101-24,115,242
2-6,25,62,123,214	27-37,98,189	52-12,73,164,291	77-48,139,266	102-23,114,241
3-5,24,61,122,213	28-36,97,188	53-11,72,163,290	78-47,138,265	103-22,113,240
4-23,60,121,212	29-35,96,187	54-10,162,289	79-46,137,264	104-21,112,239
5-3,22,59,120,211	30-34,95,186	55-9,70,161,288	80-45,136,263	105-20,111,238
6-2,21,58,119,210	31-33,94,185	56-8,69,160,287	81-44,135,262	106-19,110,237
7-1,20,57,118,209	32-93,184	57-7,68,159,286	82-43,134,261	107-18,109,236
8-19,56,117,208	33-31,92,183	58-6,67,158,285	83-42,133,260	108-17,108,235
9-18,55,116,207	34-30,91,182	59-5,66,157,284	84-41,132,259	109-16,107,234
10-17,54,115,206	35-29,90,181	60-4,65,156,283	85-40,131,258	110-15,106,233
11-16,53,114,205	36-28,89,180	61-3,64,155,282	86-39,130,257	111-14,105,232
12-15,52,113,204	37-27,88,179	62-2,63,154,281	87-38,129,256	112-13,104,231
13-14,51,112,203	38-26,87,178	63-1,62,153,280	88-37,128,255	113-12,103,230
14-13,50,111,202	39-25,86,177	64-61,152,279	89-36,127,254	114-11,102,229
15-12,49,110,201	40-24,85,176,	65-60,151,278	90-35,126,253	115-10,101,228
16-11,48,109,200	41-23,84,175,	66-59,150,277	91-34,125,252	116-9,100,227
17-10,47,108,199	42-22,83,174,	67-58,149,276	92-33,124,251	117-8,99,226
18-9,46,107,198	43-21,82,173,	68-57,148,275	93-32,123,250	118-7,98,225
19-8,45,106,197	44-20,81,172,	69-56,147,274	94-31,122,249	119-6,97,224
20-7,44,105,196	45-19,80,171,	70-55,146,273	95-30,121,248	120-5,96,223
21-6,43,104,195	46-18,79,170,	71-54,145,272	96-29,120,247	121-4,95,222
22-5,42,103,194	47-17,78,169,296	72-53,144,271	97-28,119,246	122-3,94,221
23-4,41,102,193	48-16,77,168,295	73-52,143,270	98-27,118,245	123-2,93,220
24-3,40,101,192	49-15,76,167,294	74-51,142,269	99-26,117,244	124-1,92,219
25-2,39,100,191	50-14,75,166,293	75-50,141,268	100-25,116,243	125-91,218

126-90,217	151-65,192	176-40,167	201-15,142	226-117,286	251-92,261	276-67,236
127-89,216	152-64,191	177-39,166	202-14,141	227-116,285	252-91,260	277-66,235
128-88,215	153-63,190	178-38,165	203-13,140	228-115,284	253-90,259	278-65,234
129-87,214	154-62,189	179-37,164	204-12,139	229-114,283	254-89,258	279-64,233
130-86,213	155-61,188	180-36,163	205-11,138	230-113,282	255-88,257	280-63,232
131-85,212	156-60,187	181-35,162	206-10,137	231-112,281	256-87	281-62,231
132-84,211	157-59,186	182-34,161	207-9,136	232-111,280	257-86,255	282-61,230
133-83,210	158-58,185	183-33,160	208-8,135	233-110,279	258-85,254	283-60,229
134-82,209	159-57,184	184-32,159	209-7,134,	234-109,278	259-84,253	284-59.228
135-81,208	160-56,183	185-31,158	210-6,133,	235-108,277	260-83,252	285-58,227
136-80,207	161-55,182	186-30,157	211-5,132,	236-107,276	261-82,251	286-57,226
137-79,206	162-54,181	187-29,156	212-4,131	237-106,275	262-81,250	287-56,225
138-78,205	163-53,180	188-28,155	213-3,130	238-105,274	263-80,249	288-55,224
139-77,204	164-52,179	189-27,154	214-2,129	239-104,273	264-79,248	289-54,223
140-76,203	165-51,178	190-26,153	215-1,128	240-103,272	265-78,247	290-53,222
141-75,202	166-50,177	191-25,152	216-127,296	241-102,271	266-77,246	291-52,221
142-74,201	167-49,176	192-24,151	217-126,295	242-101,270	267-76,245	292-51,220
143-73,200	168-48,175	193-23,150	218-125,294	243-100,269	268-75,244	293-50,219
144-72,199	169-47,174	194-22,149	219-124,293	244-99.268	269-74,243	294-49,218

145-71,198	170-46,173	195-21,148	220-123,292	245-98,267	270-73,242	295-48,217
146-70,197	171-45,172	196-20,147	221-122,291	246-97,266	271-72,241	296-47,216
147-69,196	172-44,171	197-19,146	222-121,290	247-96,265	272-71,240	
148-68,195	173-43,170	198-18,145	223-120,289	248-95,264	273-70,239	
149-67,194	174-42,169	199-17,144	224-119,288	249-94,263	274-69,238	
150-66,193	175-41,168	200-16,143	225-118,287	250-93,262	275-68,237	

TABLE 2 -- OPTIONS AND REMARKS FOR VARIOUS VALUES OF *n*

n	**options**	**Remarks**
<184	There is only one option 93 for 32	No necklace, may be single or multiple chains
<218	There is only one option 91for 125	No necklace, may be single or multiple chains
<296	There is only one option 127 for 216	No necklace, may be single or multiple chains
<473	There is only one option 256 for 87	No necklace, may be single or multiple chains
473	Every number has more than 1 options	May be single or multiple Necklaces

(A) n =300

List of Ts

(43,300,212)	(71,272,240)	(99,244,268)	(127,216,296)	(37,179,164)	(65,151,192)
(44,299,213)	(72,271,241)	(100,243,269)	(10,206,137),	(38,178,165)	(66,150,193)
(45,298,214)	(73,270,242)	(101,242,270)	(11,205,138)	(39,177,166)	(67,149,194)
(46,297,215)	(74,269,243),	(102,241,271)	(12,204,139)	(40,176,167)	(68,148,195)
(47,296,216)	(75,268,244)	(103,240,272)	(13,203,140)	(41,175,168)	(69,147,196)
(48,295,217)	(76,267,245)	(104,239,273)	(14,202,141)	(42,174,169)	(70,146,197)
(49,294,218)	(77,266,246)	(105,238,274)	(15,201,142)	(43,173,170),	(71,145,198)
(50,293,219)	(78,265,247)	(106,237,275)	(16,200,143),	(44,172,171)	(72,144,199)
(51,292,220)	(79,264,248)	(107,236,276)	(17,199,144)	(45,171,172)	(73,143,200)
(52,291,221)	(80,263,249)	(108,235,277)	(18,198,145)	(46,170,173)	(74,142,201)
(53,290,222)	(81,262,250)	(109,234,278)	(19,197,146)	(47,169,174)	(75,141,202)
(54,289,223),	(82,261,251)	(110,233,279)	(20,196,147)	(48,168,175)	(76,140,203)
(55,288,224)	(83,260,252)	(111,232,280)	(21,195,148)	(49,167,176)	(77,139,204)
(56,287,225)	(84,259,253)	(112,231,281)	(22,194,149)	(50,166,177)	(78,138,205)
(57,286,226)	(85,258,254)	(113,230,282)	(23,193,150)	(51,165,178)	(79,137,206)
(58,285,227)	(86,257,255)	(114,229,283)	(24,192,151)	(52,164,179)	(80,136,207)
(59,284,228),	(87,256)	(115,228,284)	(25,191,152)	(53,163,180)	(81,135,208)
(60,283,229)	(88,255,257)	(116,227,285)	(26,190,153)	(54,162,181)	(82,134,209)
(61,282,230)	(89,254,258)	(117,226,286)	(27,189,154)	(55,161,182)	(83,133,210)
(62,281,231)	(90,253,259)	(118,225,287)	(28,188,155)	(56,160,183,	(84,132,211)
(63,280,232)	(91,252,260)	(119,224,288)	(29,187,156)	(57,159,184)	(85,131,212)
(64,279,233)	(92,251,261)	(120,223,289)	(30,186,157,	(58,158,185)	(86,130,213)
(65,278,234),	(93,250,262)	(121,222,290)	(31,185,158)	(59,157,186)	(87,129,214)
(66,277,235)	(94,249,263)	(122,221,291)	(32,184,159)	(60,156,187)	(88,128,215)
(67,276,236)	(95,248,264)	(123,220,292)	(33,183,160)	(61,155,188)	(89,127,216)
(68,275,237)	(96,247,265)	(124,219,293)	(34,182,161)	(62,154,189)	(90,126,217)
(69,274,238)	(97,246,266)	(125,218,294)	(35,181,162)	(63,153,190)	(91,125,218)
(70,273,239)	(98,245,267)	(126,217,295)	(36,180,163)	(64,152,191)	(93,32,184)

There is a single choice 87 for 256. Hence it will not yield a necklace.
Eliminating the common numbers from Ts, we get

(86,257,255,88,128,215,297,46,170,173,43,300,212,131,
85,258,254,89,127,216,296,87,256)
(44,172,171,45,298,214,129,87)

(42,174,169,47)
(41,175,168,48,295,217,126,90,253,259,84,132,211)
(40,176,167,49,294,218,125,91,252,260,83,133,210)
(39,177,166,50,293,219,124,92,251,261,82,134,209)
(38,178,165,51,292,220,123,93,250,262,81,135,208)
(37,179,164,52,291,221,122,94,249,263,80,136,207)
(36,180,163,53,290,222,121)
(35,181,162,54,289,223,120)
(34,182,161,55,288,224,119)
(33,183,160,56,287,225,118)
(93,32,184,159,57,286,226,117)
(31,185,158,58,285,227,116)
(30,186,157,59,284,228,115)
(29,187,156,60,283,229,114)
(28,188,155,61,282,230,113)
(27,189,154,62,281,231,112)
(26,190,153,63,280,232,111)
(25,191,152,64,279,233,110)
(24,192,151,65,278,234,109)
(23,193,150,66,277,235,108)
(22,194,149,67,276,236,107)
(21,195,148,68,275,237,106)
(20,196,147,69,274,238,105)
(19,197,146,70,273,239,104)
(18,198,145,71,272,240,103)
(17,199,144,72,271,241,102)
(16,200,143,73,270,242,101)
(15,201,142,74,269,243,100)
(14,202,141,75,268,244,99)
(13,203,140,76,267,245,98)
(12,204,139,77,266,246,97)
(11,205,138,78,265,247,96)
(10,206,137,79,264,248,95)

Attaching the LONs around the various Ss, we get

{(213,130,86,257,255,88,128,215,297,46,170,173,43,300,212,131,
85,258,
254,89,127,216,296,87,129,214,298,45,171,172,44,299,213}
{256}
{(42,174,169,47)}
{(41,175,168,48,295,217,126,90,253,259,84,132,211)}
{(40,176,167,49,294,218,125,91,252,260,83,133,210)}
{(39,177,166,50,293,219,124,92,251,261,82,134,209)}
{(38,178,165,51,292,220,123,93,250,262,81,135,208)}
{(37,179,164,52,291,221,122,94,249,263,80,136,207)}
{(36,180,163,53,290,222,121)}
{(35,181,162,54,289,223,120)}
{(34,182,161,55,288,224,119)}
{(33,183,160,56,287,225,118)}
{(93,32,184,159,57,286,226,117)}
{(31,185,158,58,285,227,116)}
{(30,186,157,59,284,228,115)}
{(29,187,156,60,283,229,114)}
{(28,188,155,61,282,230,113)
{(27,189,154,62,281,231,112)
{1,(26,190,153,63,280,232,111)
{2,(25,191,152,64,279,233,110)
{3,(24,192,151,65,278,234,109)
{4,(23,193,150,66,277,235,108)
{5,(22,194,149,67,276,236,107)
{6,(21,195,148,68,275,237,106)
{(27,20,196,147,69,274,238,105)}
{8,(19,197,146,70,273,239,104)}
{9,(18,198,145,71,272,240,103)}
{(17,199,144,72,271,241,102)}
{(16,200,143,73,270,242,101)}
{(15,201,142,74,269,243,100)}
{(14,202,141,75,268,244,99)}
{(13,203,140,76,267,245,98)}
{(12,204,139,77,266,246,97)}
{(11,205,138,78,265,247,96)}
{(10,206,137,79,264,248,95)}

Interconnecting the Ss, we get

1) {(130,86,257,255,88,128,215,297,46,170,173,43,300,212,131,85,258,254,89,127,216,296),(47,169,174,42)}
2) {(108,235,277,66,150,193,23),(41,175,168,48,295,217,126,90,253,259,84,132,211),5,(22,194,149,67,276,236,107),(109,234,278,65,151,192,24),3,(213,299,44,172,171,45,298,214,129,87,256)}
3) (40,176,167,49,294,218,125,91,252,260,83,133,210),6,(21,195,148,68,275,237,106),(110,233,279,64,152,191,25),2}
4) {(39,177,166,50,293,219,124,92,251,261,82,134,209, 7,(20,196,147,69,274,238,105),(111,232,280,63,190, 26), (38,178,165,51,292,220,123,93,250,262,81,135,208),8,(19,197,146,70,273,239,104),(112,231,281,62,154,189,27),(37,179,164,52,291,221,122,94,249,263, 80,136,207),9,(18,198,145,71,272,240,103),(113,230,282,61,155,188,28),(36,180,163,53,290,222,121),95,248,264,79,137,206,10,17,199,144,72,271,241,102),114,229,283,60,156,187,29),(35,181,162,54,289,223,120),(96,247,265,78,138,205,11),(16,200,143,73,270,242,101),(115,228,284,59,157,186,30),(34,182,161,55,288,224,119),(97,246,266,77,139,204,12),(15,201,142,74,269,243,100),(116,227,285,58,185,31),(33,183,160,56,287,225,118),(98,245,267,76,140,203,13),14,202,141,75,268,244,99),(117,226,286,57,159,184,32,93)}
5) {1},
6) {4}

There are 4 chains and 2 isolated numbers shown in Fig. 1.

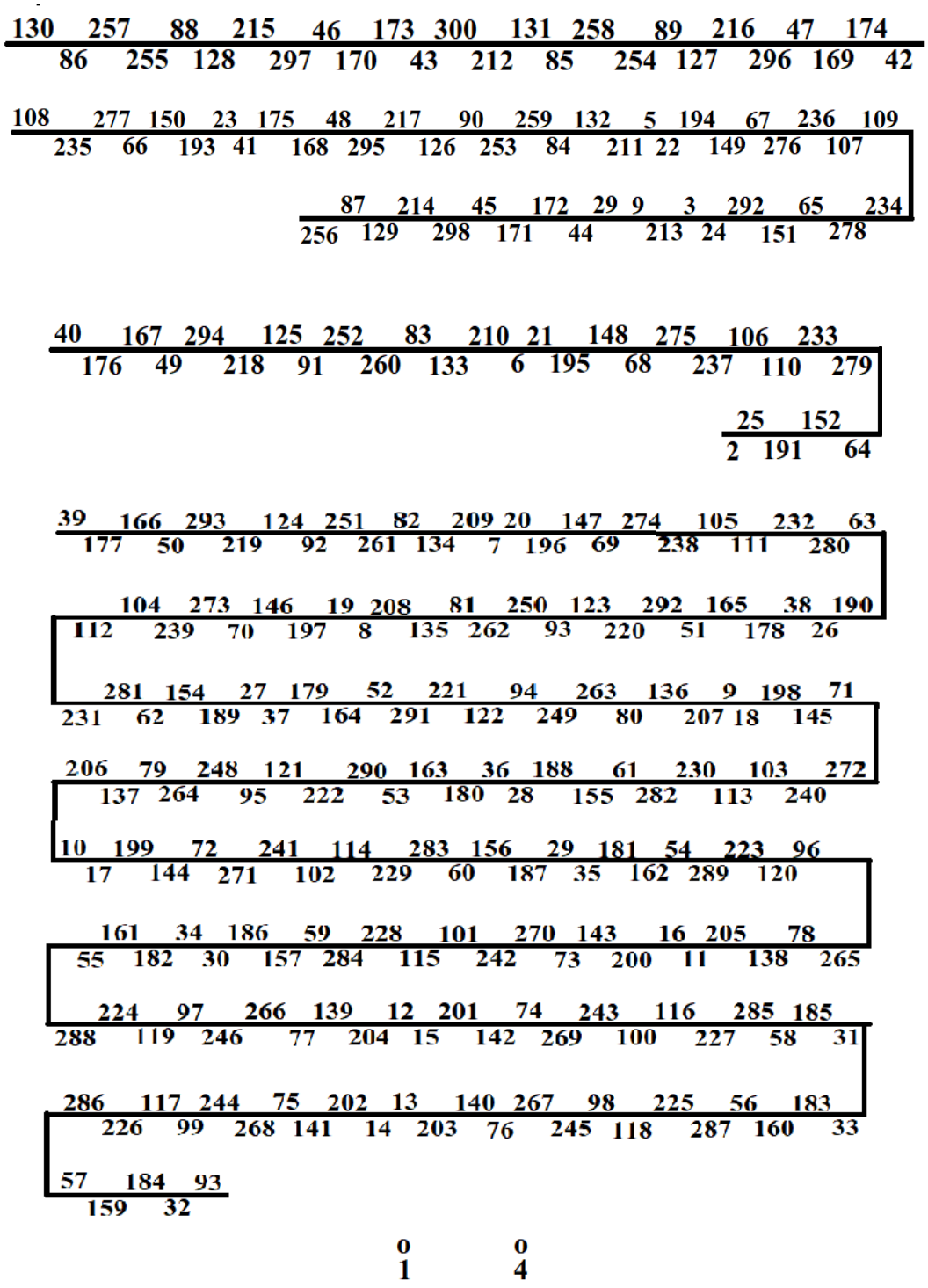


Figure 1. Chain-necklace diagram for $n = 300$.

(B) $n = 305$

List of Ts

(38,305,207) (39,304,208) (40,303,209) (41,302,210) (42,301,211) (43,300,212) (44,299,213) (45,298,214) (46,297,215) (47,296,216) (48,295,217) (49,294,218) (50,293,219) (51,292,220) (52,291,221) (53,290,222) (54,289,223), (55,288,224) (56,287,225) (57,286,226) (58,285,227) (59,284,228), (60,283,229) (61,282,230) (62,281,231) (63,280,232) (64,279,233) (65,278,234), (66,277,235)

(67,276,236) (68,275,237) (69,274,238) (70,273,239) (71,272,240) (72,271,241) (73,270,242) (74,269,243), (75,268,244) (76,267,245) (77,266,246) (78,265,247) (79,264,248) (80,263,249) (81,262,250) (82,261,251) (83,260,252) (84,259,253) (85,258,254) (86,257,255) (87,256) (88,255,257) (89,254,258) (90,253,259) (91,252,260) (92,251,261) (93,250,262) (94,249,263) (95,248,264)

(96,247,265) (97,246,266) (98,245,267) (99,244,268) (100,243,269) (101,242,270) (102,241,271) (103,240,272) (104,239,273) (105,238,274) (106,237,275) (107,236,276) (108,235,277) (109,234,278) (110,233,279) (111,232,280) (112,231,281) (113,230,282) (114,229,283) (115,228,284) (116,227,285) (117,226,286) (118,225,287) (119,224,288) (120,223,289) (121,222,290) (122,221,291) (123,220,292) (124,219,293)

(125,218,294) (126,217,295) (127,216,296) (10,206,137), (11,205,138) (12,204,139) (13,203,140) (14,202,141) (15,201,142) (16,200,143), (17,199,144) (18,198,145) (19,197,146) (20,196,147) (21,195,148) (22,194,149) (23,193,150) (24,192,151) (25,191,152) (26,190,153) (27,189,154) (28,188,155) (29,187,156) (30,186,157, (31,185,158) (32,184,159) (33,183,160) (34,182,161) (35,181,162)

(36,180,163) (37,179,164) (38,178,165) (39,177,166) (40,176,167) (41,175,168) (42,174,169) (43,173,170, (44,172,171) (45,171,172) (46,170,173) (47,169,174) (48,168,175) (49,167,176) (50,166,177) (51,165,178) (52,164,179) (53,163,180) (54,162,181) (55,161,182) (56,160,183, (57,159,184) (58,158,185) (59,157,186) (60,156,187) (61,155,188) (62,154,189) (63,153,190) (64,152,191)

(65,151,192) (66,150,193) (67,149,194) (68,148,195) (69,147,196) (70,146,197) (71,145,198) (72,144,199) (73,143,200) (74,142,201) (75,141,202) (76,140,203) (77,139,204) (78,138,205) (79,137,206) (80,136,207) (81,135,208) (82,134,209) (83,133,210) (84,132,211) (85,131,212) (86,130,213) (87,129,214) (88,128,215) (89,127,216) (90,126,217) (91,125,218) (93,32,184)

256 has only one choice 87. Hence there cannot be a necklace. An end number of a T may also have an open end. Example (264,248,95), 95 has only one option 248.

Interconnecting the trios

1. 256, 87, 129, 214, 298, 45, 171, 172, 44, 299, 213, 130, 86, 257, 255, 88, 128, 215, 297, 46, 170, 173, 43 , 300, 212, 131, 85, 258, 254, 89, 127, 216, 296, 47, 169, 174, 42, 301, 211, 132, 84, 259, 253, 90, 126, 217, 295, 48, 168, 175, 41, 302, 210, 133, 83, 260, 252, 91, 125, 218, 294, 49, 167, 176, 40, 303, 209, 134, 82, 261, 251, 92, 33, 183, 160, 56, 287, 225, 118, 98, 245, 267, 76, 140, 203, 13, 14, 202, 141, 75, 268, 244, 99,26, 190, 153, 63, 280, 232, 111, 105, 238, 274, 69, 147, 196, 20
2. 124,219, 293, 50, 166, 177, 39, 304,208, 135, 81 262 250 93,32,184,159,57,286,226,117
3. 19,197,146,70,273,239,104,112,231,281,62,154,189,27,37,179,164,52,291,221,122
4. 22,194,149,67,276,236,107,109,234,278,65,151,192,24,101,242,270,73,143,200,16,11,205,138,78,265,247,96,120,223,289,54,162,181, 35,29,187,156,60,283,229,114,102,241,271,72,144,199,17,108,235,277,66,150,193,23
5. 121,222,290,53,163,180,36,28,188,155,61,282,230,113,103,240,272,71,145,198,18
6. 116,227,285,58,158,185,31,94,249,263,80,136,207, 305,38,178,165,51,292,220,123
7. 21,195,148,68,275,237,106,110,233,279,64,152,191,25,100,243,269,74,142,201,15,12,204,139,77,266,246,97,119,224,288,55,161,182, 34,30,186,157,59,284,228,115,10,206,137,79,264,248,95

Inserting the LONs

1. (256, 87, 129, 214, 298, 45, 171, 172, 44, 299, 213, 130, 86, 257, 255, 88, 128, 215, 297, 46, 170, 173, 43 , 300, 212, 131, 85, 258, 254, 89, 127, 216, 296, 47, 169, 174, 42, 301, 211, 132, 84, 259, 253, 90, 126, 217, 295, 48, 168, 175, 41, 302, 210, 133, 83, 260, 252, 91, 125, 218, 294, 49, 167, 176, 40, 303, 209, 134, 82, 261, 251, 92, 33, 183, 160, 56, 287, 225, 118, 98, 245, 267, 76, 140, 203, 13, 14, 202, 141, 75, 268, 244, 99,26, 190, 153, 63, 280, 232, 111, 105, 238, 274, 69, 147, 196, 20),7
2. 1,(124, 219, 293, 50, 166, 177, 39, 304,208, 135, 81 262 250 93,32,184,159,57,286,226,117)
3. 8,(19,197,146,70,273,239,104,112,231,281,62,154,189,27,37,179,164,52,291,221,122),3
4. 5,(22,194,149,67,276,236,107,109,234,278,65,151,192,24,101,242,270,73,143,200,16,11,205,138,78,265,247,

96,120,223,289,54,162,181,35,29,187,156,60,283,229,114,102,241,271,72,144,199,17,108,235,277,66,150, 193,23,4

5. 121,222,290,53,163,180,36,28,188,155,61,282,230,113,103,240,272,71,145,198,18,9
6. 116,227,285,58,158,185,31,94, 249,263,80,136,207, 305,38,178,165,51,292,220,123,2
7. 6,21,195,148,68,275,237,106,110,233,279,64,152,191,25,100,243,269,74,142,201,15,12,204,139, 77,266,246,97,119,224,288,55,161,182,34,30,186,157,59,284,228,115,10,206,137,79,264,248,95

After interconnecting the strings, we get the chain-necklace diagram as shown in Fig. 2.

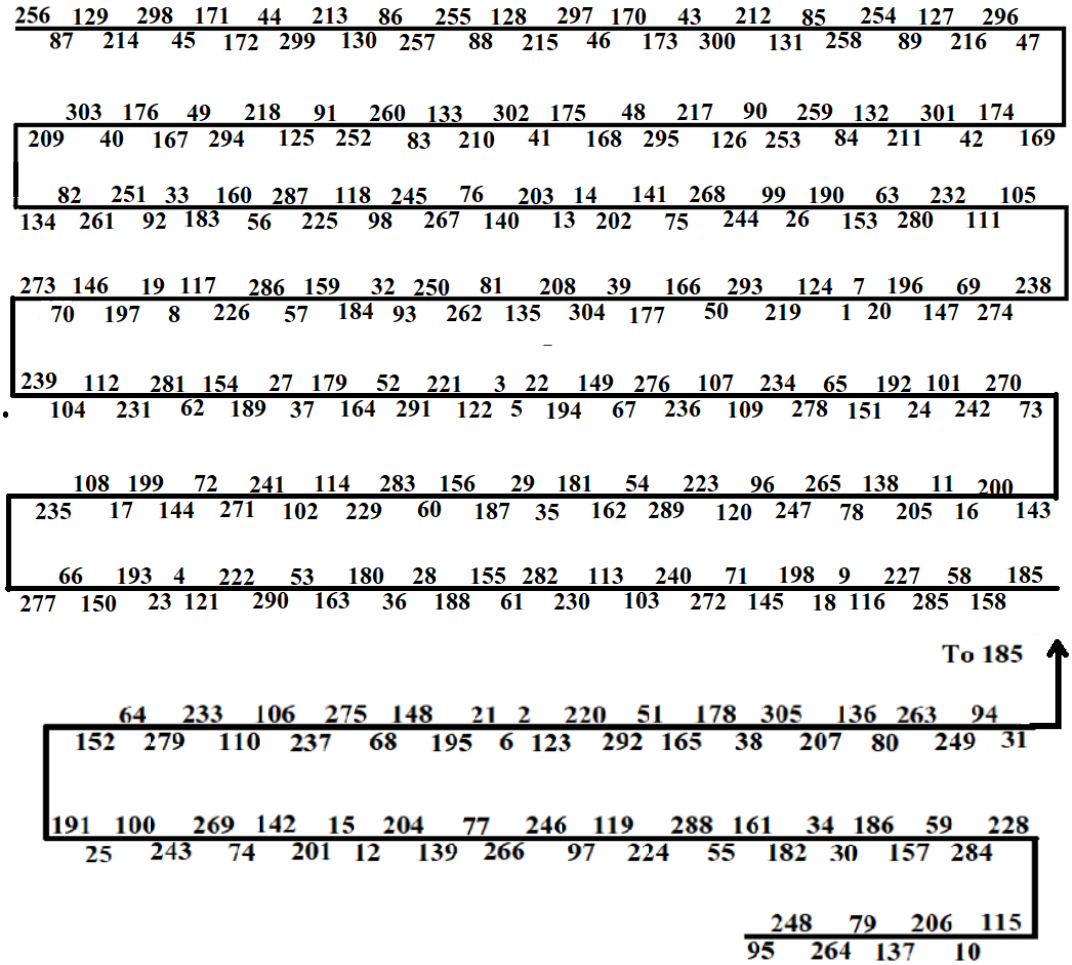


Figure 2. A single chain for n = 305.

The sum of the first number 256 and the last number 95 is not a perfect cube. Hence, it gives a full chain as shown in the Figure. This is the same reported in [2]. If we increase the number n from 305 to 472, the number 256 will continue to have a single choice 87. Hence no necklace.

(C) n = 386
Ts are:

87,256	80,263,249	73,270,242	66,277,235	59,284,228	52,291,221
86,257,255	79,264,248	72,271,241	65,278,234	58,285,227	51,292,220
85,258,254	78,265,247	71,272,240	64,279,233	57,286,226	50,293,219
84,259,253	77,266,246	70,273,239	63,280,232	56,287,225	49,294,218
83,260,252	76,267,245	69,274,238	62,281,231	55,288,224	48,295,217
82,261,251	75,268,244	68,275,237	61,282,230	54,289,223	47,296,216
81,262,250	74,269,243	67,276,236	60,283,229	53,290,222	46,297,215

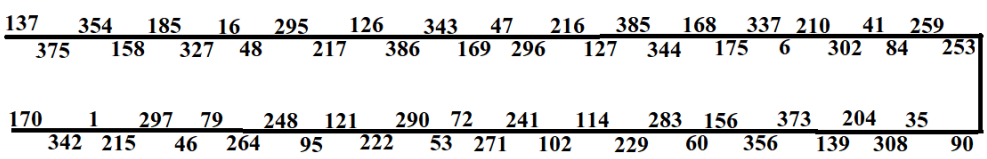
45,298,214	23,320,192	1,342,170	148,364,365	126,386,343	106,237,275
44,299,213	22,321,191	169,343,386	147,365,364	127,216,296	105,238,274
43,300,212	21,322,190	168,344,385	146,366,363	126,217,295	104,239,273
42,301,211	20,323,189	167,345,384	145,367,362	125,218,294	103,240,272
41,302,210	19,324,188	166.346,383	144,368,361	124,219,293	102,241,271
40,303,209	18,325,187	165,347,382	143,369,360	123,220,292	101,242,270
39,304,208	17,326,186	164,348,381	142,370,359	122,221,291	100,243,269
38,305,207	16,327,185	163,349,380	141,371,358	121,222,290	99,244,268
37,306,206	15,328,184	162,350,379	140,372,357	120,223,289	98,245,267
36,307,205	14,329,183	161,351,378	139,373,356	119,224,288	97,246,266
35,308,204	13,330,182	160,352,377	138,374,355	118,225,287	96,247,265
34,309,203	12,331,181	159,353,376	137,375,354	117,226,286	95,248,264
33,310,202	11,332,180	158,354,375	136,376,353	116,227,285	94,249,263
32,311,201	10,333,179	157,355,374	135,377,352	115,228,284	93,250,262
31,312,200	9,334,178	156,356,373	134,378,351	114,229,283	92,251,261
30,313,199	8,335,177	155,357,372	133,379,350	113,230,282	91,252,260
29,314,198	7,336,176	154,358,371	132,380,349	112,231,281	90,253,259
28,315,197	6,337,175	153,359,370	131,381,348	111,232,280	89,254,258
27,316,196	5,338,174	152,360,369	130,382,347	110,233,279	88,255,257
26,317,195	4,339,173	151,361,368	129,383,346	109,234,278	
25,318,194	3,340,172	150,362,367	128,384,345	108,235,277	
24,319,193	2,341,171	149,363,366	127,385,344	107,236,276	

Eliminating first the common numbers from Ts, and then attaching the LONs around them, we get

1. (137),(375,354,158), (185,327,16),48,295,217),(126,386,343),(169),(47,296,216),(127,385,344),(168),
   175,337,6,210,302,41,84,259,253,(90),35,308,204,139,373,356,(156,)(60,283,229),(114),(102,241,271),
   (72),(53,290,222),(121),(95,248,264),(79),(46,297,215),(1,342,170),
2. (77,266,246),(97),(119,224,288),(55),(70,273,239)
3. (200,312,31),(33,310,202),(141,371,358)(154),(62,281,231),(112),(104),(21,322,190),(26,317,195),
4. (37,306,206),(10,333,179) (N)
5. (256,87),(38,305,207),(136,376,353),(159),(184,328,15),(201,311,32),(93,250,262.81),44,299,213,130,382,
   347,(165),(178,334,9),(18,325,187),(29,314,198),(145,367,362),150,(193,319,24),(192,320,23),(4,339,173),
   (43,300,212),131,381,348),(164),(52,291,221),(122),(94,249,263),(80),(45,298,214),129,383,346,(166),
   (177,335,8),(19,324,188),(28,315,197),(146,366,363),(149),(194,318,25),(191,321,22),(103,240,272),(71),
   (54,289,223),(120),(96,247,265),(78),(138,374,355),(157),(186,326,17),(108,235,277),(66),(59,284,228),
   (115),(101,242,270),73),(143,369,360),(152),(64,279,233),(110),(106,237,275),(68),(148,364,365),(147),
   (196,316,27),(189,323,20),(105,238,274),(69),(56,287,225),(118),(98,245,267),(76),(49,294,218),(125),
   (91,252,260),(83),(133,379,350),(162),(181,331,12),(113,230,282),(61),(155,357,372),(140),(203,309,34),
   (30,313,199),(144,368,361),(151),(65,278,234),(109),(107,236,276),(67),(58,285,227),(116),(100,243,269),
   (74),(142,370,359),(153),(63,280,232),(111),(14,329,183),(160,352,377),135),(208,304,39),(86,257,255),
   (88),(128,384,345),(167),(176,336,7),(57,286,226),(117),(99,244,268),(75),(50,293,219),(124),(92,251,261),(82),(134,378,351),(161),(1
   82,330,13),(51,292,220),(123),(2,341,171),(172,340,3),(5,338,174),(42,301,211),
   (132,380,349),(163),(180,332,11),(205,307,36),(89,254,258,85),(40,303,209)

LONs: none.

Thus, we see that there are 4 chains and 1 necklace shown in Fig. 5. Hence, the minimal number is 305 for a single chain given in [4] and not 386 as claimed by Anurag Sahay [4]. He has given the arrangement of few numbers which tallies with our numbers also. But after adding the subsequent numbers 258,85,40,303,209, it comes to a dead end.



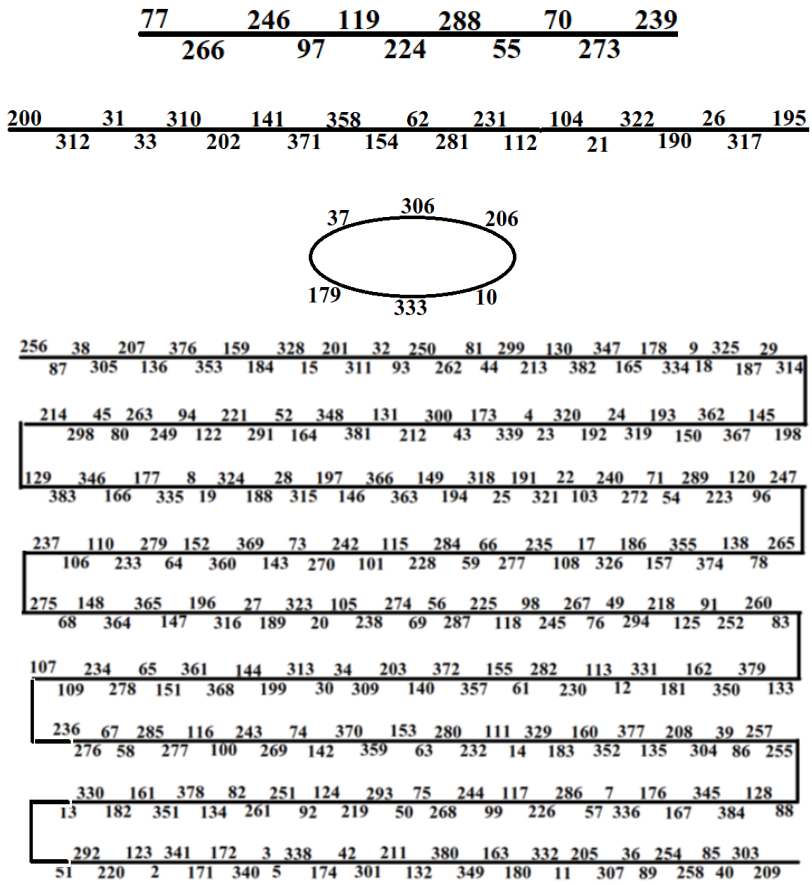


Figure 3. Chain-necklace diagram for n = 386.

(D) n = 473

Every number has more than one options. Therefore, it might give a necklace.

List of Ts

127,216,296	105,238,274	76,436,293	54,458,271	120,392,337	98,414,315
126,217,295	104,239,273	75,437,292	53,459,270	119,393,336	97,415,314
125,218,294	103,240,272	74,438,291	52,460,269	118,394,335	96,416,313
124,219,293	102,241,271	73,439,290	51,461,268	117,395,334	95,417,312
123,220,292	101,242,270	72,440,289	50,462,267	116,396,333	94,418,311
122,221,291	100,243,269	71,441,288	49,463,266	115,397,332	93,419,310
121,222,290	99,244,268	70,442,287	48,464,265	114,398,331	92,420,309
120,223,289	98,245,267	69,443,286	47,465,264	113,399,330	91,421,308
119,224,288	97,246,266	68,444,285	46,466,263	112,400,329	90,422,307
118,225,287	96,247,265	67,445,284	45,467,262	111,401,328	89,423,306
117,226,286	95,248,264	66,446,283	44,468,261	110,402,327	88,424,305
116,227,285	94,249,263	65,447,282	43,469,260	109,403,326	87,425,304
115,228,284	93,250,262	64,448,281	42,470,259	108,404,325	86,426,303
114,229,283	92,251,261	63,449,280	41,471,258	107,405,324	85,427,302
113,230,282	91,252,260	62,450,279	40,472,257	106,406,323	84,428,301
112,231,281	90,253,259	61,451,278	39,473,256	105,407,322	83,429,300
111,232,280	89,254,258	60,452,277	126,386,343	104,408,321	82,430,299
110,233,279	88,255,257	59,453,276	125,387,342	103,409,320	81,431,298
109,234,278	87,256,473	58,454,275	124,388,341	102,410,319	80,432,297
108,235,277	79,433,296	57,455,274	123,389,340	101,411,318	169,343,386
107,236,276	78,434,295	56,456,273	122,390,339	100,412,317	168,344,385
106,237,275	77,435,294	55,457,272	121,391,338	99,413,316	

Eliminating the common numbers from Ts

{127,216,296, 433,79,46,466,263,249,94,418,311}
{169,343,386,126,217,295,434,78,47,465,264,248,95,417,312}
{342,387,125,218,294,435,77,48,464,265,247,96,416,313}
{341,388,124,219,293,436,76,49,463,266,246,97,415,314}
{340,389,123,220,292,437,75,50,462,267,245,98,414,315}
{339,390,122,221,291,438,74,51,461,268,244,99,413,316}
{338,391,121,222,290,439,73,52,460,269,243,100,
412,317}
{337,392,120,223,289,440,72,53,459,270,242,101,
411,318}
{336,393,119,224,288,441,71,54,458,271,241,102,
410,319}
{335,394,118,225,287,442,70,55,457,272,240,103,
409,320}
{334,395,117,226,286,443,69,56,456,273,239,104,
408,321}
{333,396,116,227,285,444,68,57,455,274,238,105,
407,322}
{332,397,115,228,284,445,67,58,454,275,237,106,
406,323}
{331,398,114,229,283,446,66,59,453,276,236,107,
405,324}
{330,399,113,230,282,447,65,60,452,277,235,108,
403,325}
{329,400,112,231,281,448,64,61,451,278,234,109,
402,326}
{328,401,111,232,280,449,63,62,450,279,233,110,
401,327}
{310,419,93,250,262,467,45}
{309,420,92,251,261,468,44}
{308,421,91,252,260,469,43}
{307,422,90,253,259,470,42}
{306,423,89,254,258,471,41}
{305,424,88,255,257,472,40}
{304,425,87,256,456,473,39}
{86,426,303}
{85,427,302}
{84,428,301}
{83,429,300}
{82,430,299}
{81,431,298}
{80,432,297}
{168,344,385}

LONs are
1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,2
6,27,28,29,30,31,32,33,34,35,36,37,38,128,129,130,131,132,133,13
4,135,136,137,138,139,140,141,142,143,144,145,146,147,148,149,
150,151,152,153,
154,155,156,157,158,159,160,161,162,163,164,165,166,167,168,16
9,170,171,172,173,174,175,176,177,178,
179,180,181,182,183,184,185,186,187,188,189,190,191,192,193,19
4,195,196,197,198,199,200,201,202,203,
204,205,206,207,208,209,210,211,212,213,214,215,345,346,347,34
8,349,350,351,352,353,354,355,356,357,
358,359,360,361,362,363,364,365,366,367,368,369,370,371,372,37
3,374,375,376,377,378,379,380,381,382,
383,384,407,420,430,455,468

Inserting the LONs, we get

{168,344,385,127,216,296, 433,79,46,466,263,249,94,418,311}
{217,295,434,78,47,465,264,248,95,417,312,200,16,11,205,138,374,355,157,186}
{170,173,342,387,125,218,294,435,77,48,464,265,247,96,416,313}
{172,171,341,388,124,219,293,436,76,49,463,266,246,97,415,314,29,35,181,162,350,379,133}
{340,389,123,220,292,437,75,50,462,267,245,98,414,315,197,19,8}
{339,390,122,221,291,438,74,51,461,268,244,99,413,316,27,189,154,358,371,141,202,14,13,203,309,420,92,251,261,468,
44,81,431,298,214,129,383,346,166,177}
{126,386,343,169,174,338,391,121,222,290,439,73,52,460,269,243,100,412,317,195,21}
{175,337,392,120,223,289,440,72,53,459,270,242,101,411,318}
{176,336,393,119,224,288,441,71,54,458,271,241,102,410,319}
{335,394,118,225,287,442,70,146,366,363,149}
{320,409,103,240,272,457,55,161,351,378,134,82,430,299,213,3,24,192,151,361,368,144,199,17,10,206,137,375,354,158,
{334,395,117,226,286,443,69,56,456,273,239,104,408,321,191,25,2,6,210}

{37,179,333,396,116,227,285,444,68, 57,455,274,238,105,407,322}
{194,22,5,211,132,380,349,163,180,332,397,115,228,284,445,67,58,454,275,237,106,406,323,20,196,147,365,364,148}
398,331,12,15,201,142,370,359,153,190,322,407,105,238,274,455,57,159,353,376,136,207,9,18,198,145,367,362,150,193
114,229,283,446,66,59,453,276,236,107,405,324}
{30,34,182,330,399,113,230,282,447,65,60,452,277,235,108,404,325}
{39,473,456,256,87,425,304,208,135,377,352,160,183,329,400,112,231,281,448,64,152,360,369,143}
{140,372,357,155,61,451,278,234,109,402,326}
{32,184,328,401,111,232,280,449,63,62,450,279,233,110,401,327}
{185,31,33,310,419,93,250,262,467,45}
{187,156,356,373,139,204,308,421,91,252,260,469,43}
188,28,36,307,422,90,253,259,470,42,301,428,84}
{306,423,89,254,258,471,41,85,427,302,131,381,348,164)}

{178,165,347,382,130,86,426,303}
{209,7,1,26,38,305,424,88,255,257,472,40}
{83,429,300,212,4,23}
{80,432,297,215,128,384,345,167}

After interconnecting the Ss, we get the chain-necklace diagram as shown in Fig. 4.

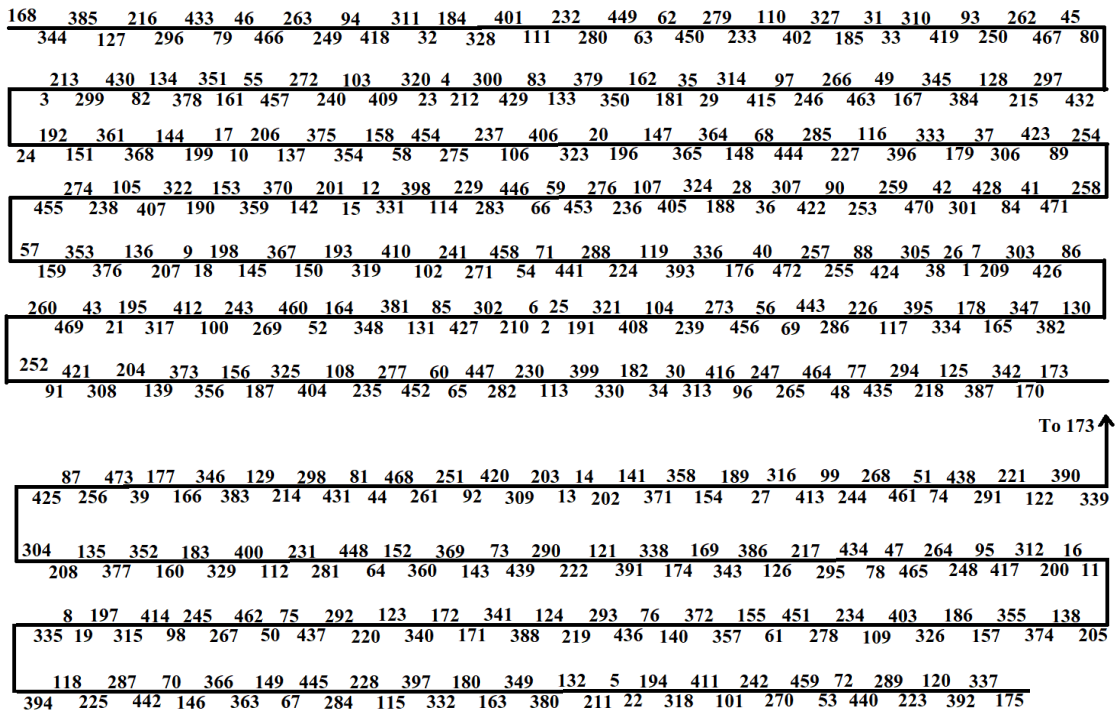


Figure 4. Chain-necklace diagram for $n = 473$.

The sum of the first number 168 and the last number 175 is 343; a perfect cube. Hence, it is a full necklace. This is the same as given by Giovanni [4]. It has been verified that the numbers below 473 do not form a full necklace.

III CONCLUSION

Rathore's method is used for obtaining the chain-necklace diagrams such that the sum of two consecutive numbers is a perfect cube. The spread in the cube of numbers is much larger than that in square values. Hence, we get the first full chain for $n = 305$ and the first full necklace for $n = 473$ as against 15 and 32, respectively, for the square case [4].

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[3] T. S. Rathore, "Arranging integer numbers on a loop such that the sum of any two adjacent numbers is a perfect square", IEEE Regio 10 Symposium, IIT Bombay, July 1-3, 2022.
[4] https://www.primepuzzles.net, Puzzle 311, Sum to a cube

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