Some Properties of Integer Numbers

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Abstract -- **Some properties of the iterations of subtracting two *n*-digit numbers (one in the forward and the other in the reverse order) are brought out through a number of examples and proved. The iterations are shown in the form of a Spoon Diagram. It consists of two parts: a straight line and a loop. Therefore, no kernel exists. The number of iterations is either 5 or 2 on the loop in cyclic order. Many more such numbers can be generated by adding or subtracting specific numbers.**

Keywords: Kaprekar operation, Number theory, Kernel, Kernel number

I. INTRODUCTION

LET an *n*-digit number be represented as

$$A_n = a_1 a_2 a_{3.} \dots a_n \tag{1}$$

and

$$A_{nR} = a_n a_{n\text{-}1.} \dots a_{1.} \tag{2}$$

where A_{nR} is the same as A_n but digits are in reverse order. Then

$$A_n - A_{nR} \;=\; b_1 b_2 b_{3.} \dots b_n \tag{3}$$

When A_n represents the maximum value obtained by arranging *n* digits, the operation given by equation (3) is designated as *Kaprekar operation* [1]. However, A_n should not contain all the digits that are same, for example, 2222.

Equation (3) can be written as

$$\mathscr{K}(A_n) = (B_n) = \; b_1 b_2 b_{3.} \dots b_{n.} \tag{4}$$

If successive $\mathscr{K}(A_n)$ are carried out on the numbers generated after each iteration, and reach to a particular number from where no new number is generated. This number is called the kernel. It is known [1] that if *n* digits in N_n are arranged such that it gives the maximum value, then for $n = 3$, the kernel is 495, for $n = 4$, kernel is 6174. For some values of *n*, there may not be any kernel (n = 2, 5 and 7) or there may be more than 1 kernels (n = 6, 8). The maximum number of iterations to reach the kernel is 7 for $n = 4$.

In this paper, we derive some properties of A_n, where A_n is *any* *n*-digit number (not necessarily it should be maximum). In this case equation (4) will be written as

$$\mathscr{R}(A_n) = \;|A_n - A_{nR}|\; = B_n = b_1 b_2 b_{3.} \dots b_n \tag{5}$$

In the next section, we observe through examples some properties of integer numbers and in the following section these properties are proved. Section IV gives the computer verification of the results. Section V gives the conclusion.

II. PROPERTIES OF INTEGER NUMBERS

For convenience, let the *n*-bit number A_n be partitioned as shown in Fig. 1 where a_m is the middle digit, and

$$p = \begin{cases} \dfrac{n}{2} \\ \dfrac{(n-1)}{2}. \end{cases}$$

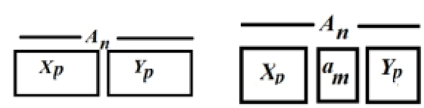


Figure 1. Partioning of A_n (*a*) *n* even (*b*) *n* odd.

Example 1: Let A_7 = 7324013, then

$\mathscr{R}$(7324013) = |7324013 – 3104237| = 4219776

$\mathscr{R}$(4219776) = |4219776 -6779124| = 2559348. Continuing the same procedure, we get the sequence of numbers as

5880204, 1859319, 7280262, 4659435, 0690129, 8520831, 7140573, 3390156, 3120777, 4649436, 1700028, 6500053, 3099987, 4799916, 1400058, 7099983, 3200076, 3499947, 3999996, 2999997, 4999995, 0999999, 8999991, 6999993, 2999997.

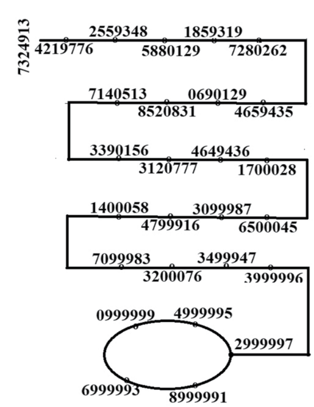


Figure 2. SD for Example 1.

Now the repetition starts after every 5 iterations. The successive values of $\mathscr{R}(A_7)$ are shown in the form of a spoon diagram (SD) in Fig. 2. It is named SD because of its shape. It consists of a line and a loop. The process goes from left to right on the line followed by clockwise rotation on the loop. The given number A_n will be referred as input to SD.

For convenience, any number on the line will be designated as NL and that on the loop as NO. Number of iterations on the line will be designated as NIL and that on the loop as NIO.

Example 2: Let A_6 = 642135.

The SD is shown in Fig. 3.

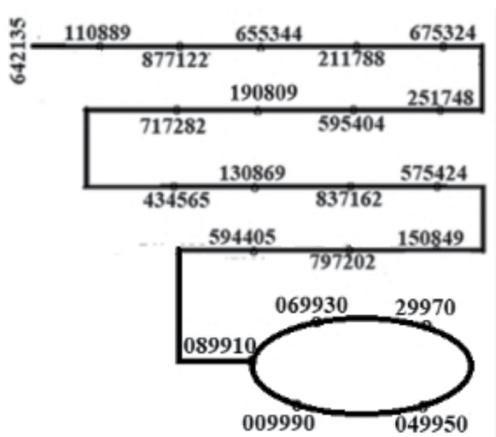


Figure 3. SD for Example 2

Example 3: Let A_5 = 64321.
The SD is shown in Fig.4.

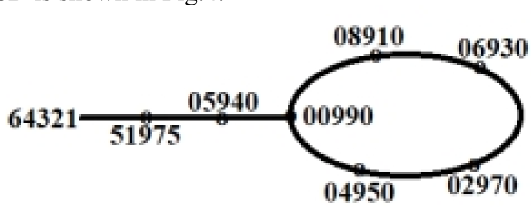
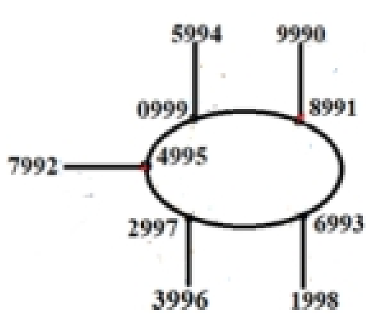
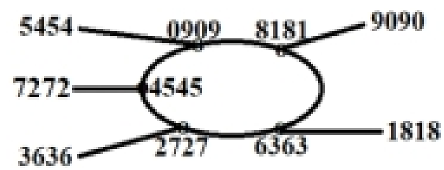


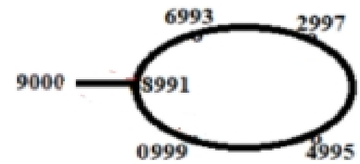
Figure 4. SD for Example 3



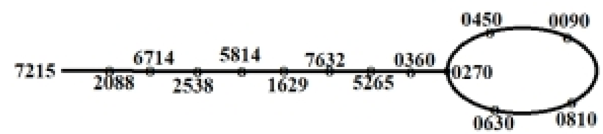
(*a*)



(*b*)



(*c*)



(*d*)

(*e*)

(*f*)

Figure 5. SDs for Example 4(*a*) - 4(*f*).

Example 4: SDs for A_4 = 7992, 5994, 9990, 1998 and 3996 are as shown in Fig. 5(*a*). All of them share the same loop. SDs for 7272, 5454, 9090, 1818 and 3636 are shown in Fig. 5(*b*). They share the same loop. SDs for A_4 = 9000, 7215, are shown in figures 5(*c*) and (*d*), respectively. SDs for A_4 = 9900 and 6402 are shown in figures 5(*e*) and (*f*), respectively. They share the same loop.

Example 5: Let A_3 = 135.
The SD is shown in Fig. 6.

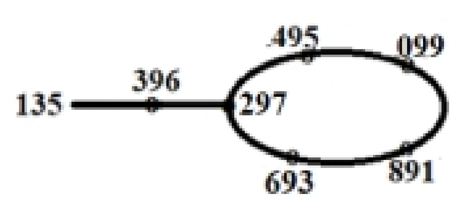


Figure 6. SD for Example 5

Example 6: Let A_2 = 86.

The SD is shown in Fig. 7.

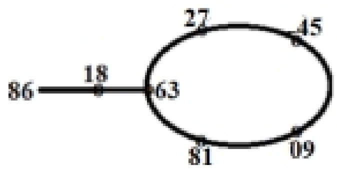


Figure 7. SD for Example 6.

From SDs shown in figures 2 to 7, we note the following properties.

1. SDs of all the figures have two parts:
(*i*) a straight line and (*ii*) a loop.

2. All NLs and NOs of an SD are distinct.

3. For the same n, but different inputs, there may be different SDs. See Fig. 5(*a*), (*b*), (*c*) and (*d*). Though the four loops are different, each has 5 NOs, and the last digits of them occur in the same cyclic order, *i.e.*, 1, 3, 7, 5, 9.

4. If (A_n) is divisible by 9, No kernel exists whatever be the input.

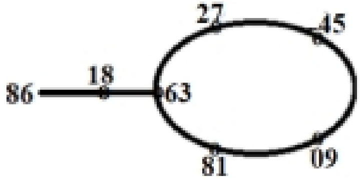
5. (*a*) If S_l is such that $S_l = S_{lR}$, is centrally located in A_n and the value of the number to the right of S_l is less than that to the left of S_l in reverse order, then $\mathscr{R}(A_n)$ will have a string R_l with all l digits 9 and centrally located.

Example: Let $A_n = 83932323927$. Then
$\mathscr{R}(83932323927) = 1099999989$. Here $S_l = 9323239$, 27 is a number to the right of S_l and 83 is a number to the left of S_l, $27 < 38$. Therefore $R_l = 999999$.

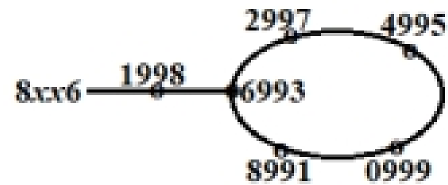
(*b*) If the value of the number to the right of S_l is more than that to the left of S_l in reverse order, then $\mathscr{R}(A_n)$ will have a string R_l with all l digits 0 centrally located.
Example: $\mathscr{R}(81932323936) = 18000000018$.

Here $S_6 = 9323239$, 36 is a number to the right of S_6, 81 is a number to the left of S_6. $36 > 18$.Therefore $R_l = 0000000$.

6. (*a*) In an input A_n (n even), l number of xs (where x is any number from 0 to 9) are inserted after the right of X_p and before the left of Y_p, the new SD will have $2l$ number of 9s in the middle of each of the numbers in the original SD.



(*a*)

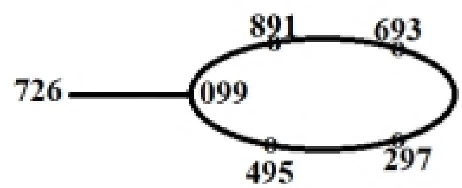


(*b*)

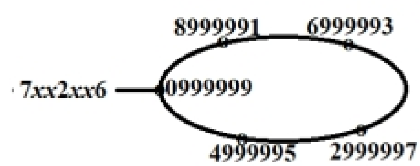
Figure 8. SD for 8xx6 where x can be any number 1 to 9.

Let $A_2 = 86$. Then inserting x on right of X_p and left of Y_p for n even, it becomes 8*xx*6. The two SDs for 86 and 8*xx*6 are shown in Figs. 7 and 8, respectively.

(*b*) In any input A_n (n odd), if l number of xs (where x is any number from 0 to 9) are inserted both on the right of X_p and left of Y_p, the new SD will have $2l$+1 number of 9s in place of a_m. Consider $A_3 = 726$. Let us make it 7*xx*2*xx*6. The SDs of the original input 726 and the modified input 7*xx*2*xx*6 are shown in Figs. 9(a) and (*b*), respectively.



(*a*)



(*b*)

Figure 9. (*a*) SDs for 726 and (*b*) 7*xx*2*xx*6.

7. Different inputs, which will give the same SD, may be possible. For example, in Fig. 5(*c*), 9004 is a possible input in place of 9000. Also, 9004 can be an input in place of 9990.

8. In $\mathscr{R}(A_n)$, we subtract the smaller number from the bigger number. Therefore, $a_n \leq a_1$. If equality holds, then $\mathscr{R}(A_n)$ will have both the first and the last digits as 0. Subsequently, all the numbers obtained after successive applications $\mathscr{R}(A_n)$ will have this property. See examples 2 and 3 above. In such cases, immediate neighbouring digits should be considered.

9. (*a*) A loop may have NOs either 5 or 2. These loops will be designated as L5 and L2, respectively.
(*b*) The sum of first and last digits of NOs in L5 is 9, and in L2 it is 10.
10. (*a*) In L-2 type, the last digits of all the NOs are even, therefore, the input must have (*i*) both a_1 and a_n of same nature

(either both odd, or both even) and (*ii*) should be reverse of one of the NOs. If these conditions do not hold, then input must be followed by a number of iterations until these conditions are met. In Fig. 4(*c*), input 9900 is connected to the loop after two iterations to give 8712 which is also present in the loop with digits reversed. Both NOs rotate in cyclic order with the last digits as 4 and 8 (both even) for different inputs.

(*b*) In L5-type, the last digits of all the NOs are odd, therefore, the input must have (*i*) both a_1 and a_n of opposite nature (if one is odd, the other has to be even) and (*ii*) should be reverse of one of the NOs. If these conditions do not hold, then input must be followed by a number of iterations until these conditions are met. In Fig. 6, input 135 is connected to the loop after one iteration to give 396 which is also present in the loop with digits reversed. In Fig. 2, several iterations are needed to fulfill the conditions. All the five NOs rotates in cyclic order with least significant digits as 1 3 7 5 9 (all odd) for different inputs.

11. Classification of inputs to the loop

(*a*) (*i*) L2-type: Inputs which yield L2 as shown in Fig. 5(*e*) and (*f*).

(*ii*) L5-type: Inputs which yield L5 as shown in all the remaining figures.

(*b*) (*i*) Inputs which give, after 2 or 5 iterations, the NOs with digits reversed as shown in all figures except in Figs. 3, 5(*c*) and 9. There are many possible numbers which will give the same SD. Therefore, only those numbers, whose digits are in reverse order of the numberswhich exist on the loop, can satisfy this property.

(*ii*) Inputs which do not give digits in reversed order after any number of iterations as shown in Figs. 3, 5(*c*) and 9.

(*c*) Inputs which have no number on the line. If any of the NOs becomes the input, then there will be no NL. As examples: 45, 63, 2178, 6534, 181, 9090, 4995, 8991.

(*d*) Inputs have $X_{pR} = Y_p$ which do not give any output. These inputs are trivial ones. Examples, 181, 2222, 321123.

12. (*a*) For *n* odd, 9 is centrally located in all NOs. See SDs of figures 2, 4, 6.

(*b*) For *n* even and the digits repeat, *i.e.*, Y_p is same as X_p, all the NOs will also have repetition of digits. See Fig. 5(*b*).

13. For the same *n*, but different numbers, there may be different NILs to reach CN (see Fig. 5), however, the NIO is either 5 or 2.

14. For $n > 2$, one possible loop can be generated by inserting in the middle (n - 2) number of *x*s. Table 1 shows the NOs for n = 2 to 7 where *x* is chosen as 9. However, there are other possible loops also. See example 2, 3.

TABLE 1-- NOS FOR DIFFERENT VALUES OF *N*

n	NOs
2	63, 27, 45, 09, 81
3	693, 297, 495, 099, 891
4	6993, 2997, 4995, 0999, 8991
5	69993, 29997, 49995, 09999, 89991
6	699993, 299997, 499995, 099999, 899991
7	6999993, 2999997, 4999995, 0999999, 8999991

III. PROOF OF PROPERTIES

Only properties 4 and 11 require proofs, rest are all obvious or trivial.

Proof for property (4)

In the operation $\mathscr{R}(A_n)$, we subtract smaller number from the bigger number. Hence

$$a_1 \geq a_n \tag{5}$$

$$\begin{aligned}\mathscr{R}(N_n) &= (a_1 a_2 a_3 \ldots a_n) - (a_n a_{n-1} \ldots a_1)\\ &= (a_1 10^{n-1} + a_2 10^{n-2} + a_3 10^{n-3} + \ldots + \; + a_{n-2} 10^2 \\ &\quad + a_{n-1} 10^1 a_n 10^0) \\ &\quad - (a_n 10^{n-1} + a_{n-1} 10^{n-2} + a_{n-2} 10^{n-3} + \ldots + \\ &\quad a_3 10^2 + a_2 10^1 + a_1 10^0)\\ &= \sum_i^n (a_i - a_{n+1-i})\,(10^{n-i} - 10^{i-1}),\\ &= \sum_i^p (a_i - a_{n+1-i})\,(10^{n-i} - 10^{i-1}),\\ &= \sum_i^p (a_i - a_{n+1-i})\,10^{i-1}(10^{n-2i+1} - 1)\end{aligned} \tag{6}$$

$$\text{where } p = \begin{cases} n/2, & n \text{ even} \\ (n+1)/2, & n \text{ odd.}\end{cases}$$

All terms in (6) will be present for *n* even, and the middle term b_m may be 0 (for $Y_p > X_p$) or 9 (for $Y_p < X_p$). The *k*th term has a factor $(10^k - 1)$ where $k = n$-1 to 0. This factor is a multiple of 9. Hence each term is a multiple of 9. Therefore, the entire expression is divisible by 9.

Proof of Property (10)

Following the rules of subtraction

$$b_1 = \begin{cases} |a_1 - a_n| - 1, & \text{if there is a borrow by by } a_2 \text{ from } a_1 \\ |a_1 - a_n|, & \text{if there is no such borrow}\end{cases} \tag{7}$$

and

$$b_n = 10 - |a_1 - a_n| \tag{8}$$

$$b_1 + b_n = \begin{cases} 9, & \text{if there is a borrow} \\ 10, & \text{no borrow}\end{cases} \tag{9}$$

There are three cases:

Case 1: In this case, condition is

$$b_1 + b_n = 9 \tag{10}$$

and is valid after each subsequent iteration. Since the sum of all the digits b_1 to b_n should be 9 (as per the property (4)) after any iteration, the sum of the digits from b_2 to b_{n-1} should also be 9. Therefore, the admissible combinations for (b_1, b_n)

$$(9, 0), (8, 1), (7, 2), (6, 3), (5, 4) \tag{11}$$

In each combination, we have taken the first number greater than the second one; it can be otherwise also. Refer to examples 4(*a*), 4(*b*) and 5. Note that all the numbers have the sum of all the digits as 9; so also sum of the first and the last digits. There are in all 5 numbers on the loop. Thus, there would be 5 groups of numbers whose CN will be one of the numbers on the loop.

It can be verified that same thing happens while starting with other b_1, b_n combinations of equation (11).

Thus, there are 5 and only 5 numbers on the loop and they are all odd. The last digits are odd and repeat in cyclic order 1,3,7,5,9 for different numbers.

Case 2: In this case

$$b_1 + b_n = 10 \tag{12}$$

and is valid after each subsequent iteration. It can easily be verified that this case does not exist for n = 2 and 3. For n > 3, the possible combinations of (b_1, b_n) are

$$(9, 1), (8, 2), (7, 3), (6, 4), (5, 5). \tag{13}$$

The last combination, will give on all the following iterations $b_1 = b_n = 0$. In this case, the neighbouring digits b_2,b_{n-1} should be considered. (See example 2 and 3).

Since $b_1 + b_n = 10$, the sum of all the remaining digits should be 8 after any iteration. The permissible combinations are-

$$(8, 0), (7, 1), (6, 2), (5, 3), (4, 4) \tag{14}$$

Let us take 9801. The SD is shown in Fig. 5(*b*). Note that all the numbers have $b_1 + b_4 = 10$. There are in all 2 numbers on the loop.

It can be verified that same thing happens with other b_1, b_4 combinations from equation (13) and b_2, b_3 from equation (14). Thus, there are 2 and only 2 numbers on the loop and both are even. They repeat in cyclic order 6534 and 2178 for successive iterations. Thus, there would be 2 groups of numbers which will follow this pattern.

Case 3: In this case, the successive iterations may turn out $b_1 + b_n = 9$ or 10. In such a case, the SD will contain L5. The occurrences of 9 or 10 is random depending upon the number A_n. Therefore, a general proof may be difficult. See Examples 1, 2 and 3. The first one has 10,10,9,10,9,9,9,9,10,9,10,1,9,9,10,10,9,10,9,10,9,9,9,9,9,9.

IV. VERIFICATION

A computer program has been written to verify the theory. Various inputs and corresponding outputs (on line and in the loop) obtained are shown in Table 2. They are exactly the same as obtained manually. They also satisfy the properties laid down.

TABLE 2 -- COMPUTER RESULTS

1. 7524513 On line: 7524513 4370256 2150478 6590034 2289078 6420744 1950498 6990093 3089097 4820706 1249578 7509843 4020786 2849418 5300064 0700029 8500041 7099983 3200076 3499947 3999996 In Loop: 2999997 4999995 0999999 8999991 6999993
2. 642135 On line: 642135 110889 877122 655344 211788 675324 251748 595404 190809 717282 434565 130869 837162 575424 150849 797202 594405 In Loop: 089910 069930 029970 049950 009990
3. 64321 On line: 64321 51975 05940 In Loop: 00990 08910 06930 02970 04950
4. 7992 On line: 7992 In Loop: 4995 0999 8991 6993 2997
5. 5994 On line: 5994 In Loop: 0999 8991 6993 2997 4995
6. 9990 On line: 9990 In Loop: 8991 6993 2997 4995 0999
7. 1998 On line: 1998 In Loop: 6993 2997 4995 0999 8991
8. 3996 On line: 3996 In Loop: 2997 4995 0999 8991 6993
9. 5454 On line: 5454 In Loop: 0909 8181 6363 2727 4545
10. 9090 On line: 9090 In Loop: 8181 6363 2727 4545 0909
11. 1818 On line: 1818 In Loop: 6363 2727 4545 0909 8181
12. 3636 On line: 3636 In Loop: 2727 4545 0909 8181 6363
13. 7272 On line: 7272 In Loop: 4545 0909 8181 6363 2727
14. 9000 On line: 9000 In Loop: 8991 6993 2997 4995 0999

15. 7215 On line: 7215 2088 6714 2538 5814 1629 7632 5265 0360 In Loop: 0270 0450 0090 0810 0630
16. 9900 On line: 9900 9801 8712 In Loop: 6534 2178
17. 6402 On line: 6402 4356 In Loop: 2178 6534
18. 135 On line: 135 396 In Loop: 297 495 099 891 693
19. 86 On line: 86 18 In Loop: 63 27 45 09 81
20. 8116 On line: 8116 1998 In Loop: 6993 2997 4995 0999 8991
21. 8886 On line: 8886 1998 In Loop: 6993 2997 4995 0999 8991
22. 726 On line: 726, In Loop: 099 891 693 297 495,
23. 7112116 On line: 7112116, In Loop: 0999999 8999991 6999993 2999997 4999995
24. 7992996 On line: 7992996 In Loop: 0999999 8999991 6999993 2999997 4999995
25. 63 On line: Nil In Loop: 63, 27, 45, 09, 81
26. 181 On line: 181 In Loop: 000

V. CONCLUSION

Some properties of the successive iterations $\mathscr{R}(N_n)$ have been brought out through a number of examples and then proved. The number of iterations is shown in the form of an SD. It consists of two parts: a straight line and a loop. Therefore, no kernel exists. Successive $\mathscr{R}(N_n)$ ends up after either 5 or 2 iterations on the loop in cyclic order. Many more such numbers can be generated by adding or subtracting specific numbers. The results have been verified by writing a computer program.

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